

## Weighted Spaces of Holomorphic Functions on Banach Spaces and the Approximation Property

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*Abstract:* In this paper, we study the linearization theorem for the weighted space  $\mathcal{H}_w(U; F)$  of holomorphic functions defined on an open subset  $U$  of a Banach space  $E$  with values in a Banach space  $F$ . After having introduced a locally convex topology  $\tau_{\mathcal{M}}$  on the space  $\mathcal{H}_w(U; F)$ , we show that  $(\mathcal{H}_w(U; F), \tau_{\mathcal{M}})$  is topologically isomorphic to  $(\mathcal{L}(\mathcal{G}_w(U); F), \tau_c)$  where  $\mathcal{G}_w(U)$  is the predual of  $\mathcal{H}_w(U)$  consisting of all linear functionals whose restrictions to the closed unit ball of  $\mathcal{H}_w(U)$  are continuous for the compact open topology  $\tau_0$ . Finally, these results have been used in characterizing the approximation property for the space  $\mathcal{H}_w(U)$  and its predual for a suitably restricted weight  $w$ .

*Key words:* Holomorphic mappings, weighted spaces of holomorphic functions, linearization, approximation property.

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### 1. INTRODUCTION

Approximation properties for various classes of holomorphic functions have been studied earlier by using linearization techniques in [6], [7], [8], [18], etc. If  $E$  and  $F$  are Banach spaces and  $U$  is an open subset of  $E$ , then the linearization results help in identifying a given class of holomorphic functions defined on  $U$  with values in  $F$ , with the space of continuous linear mappings from a certain Banach space  $G$  to  $F$ ; indeed, a holomorphic mapping is being identified with a linear operator through linearization results. This study for various classes of holomorphic mappings have been carried out by Beltran [2], Galindo, Garcia and Maestre [11], Mazet [17], Mujica [18, 19, 20] and several other mathematicians.

On the other hand, whereas the weighted spaces of holomorphic functions defined on an open subset of the finite dimensional space  $\mathbb{C}^N$ ,  $N \in \mathbb{N}$  (set of natural numbers) have been investigated in [3], [4], [5], [24], etc., the infinite dimensional case was considered by Garcia, Maestre and Rueda [12], Jorda [15], Rueda [25]. The present paper is an attempt to study approximation

properties for weighted spaces of holomorphic mappings. Indeed, after having given preliminaries in Section 2, we prove in Section 3 a linearization theorem for the weighted space  $\mathcal{H}_w(U; F)$  of holomorphic functions defined on  $U$  with values in  $F$ . As an application of this result, we show that  $E$  is topologically isomorphic to a complemented subspace of  $\mathcal{G}_w(U)$  for those weights  $w$  for which  $\mathcal{H}_w(U)$  contains all the polynomials. In case of a weight being given by an entire function with positive coefficients, we also obtain estimates for the norm of the topological isomorphism.

In Section 4 we define a locally convex topology  $\tau_{\mathcal{M}}$  on the space  $\mathcal{H}_w(U; F)$  and show the topological isomorphism between the spaces  $(\mathcal{H}_w(U; F), \tau_{\mathcal{M}})$  and  $(\mathcal{L}(\mathcal{G}_w(U); F), \tau_c)$  for a weight  $w$  on an open set  $U$ .

Finally, in Section 5 we consider the applications of results proved in Sections 3 and 4 to obtain characterizations of the approximation property for the space  $\mathcal{H}_w(U)$  and its predual  $\mathcal{G}_w(U)$ ; for instance, we prove that  $\mathcal{H}_w(U)$  has the approximation property if and only if it satisfies the holomorphic analogue of Theorem 2.4(iv), *i.e.*, for any Banach space  $F$ , each mapping in  $\mathcal{H}_w(U; F)$  with relatively compact range belongs to the  $\|\cdot\|_w$ -closure of the subspace of  $\mathcal{H}_w(U; F)$  consisting of finite dimensional holomorphic mappings. Besides, it is proved that for a suitably restricted  $w$  and  $U$ ,  $\mathcal{G}_w(U)$  has the approximation property if and only if  $E$  has the approximation property.

## 2. PRELIMINARIES

Throughout this paper, the symbols  $\mathbb{N}, \mathbb{N}_0$  and  $\mathbb{C}$  respectively denote the set of natural numbers,  $\mathbb{N} \cup \{0\}$  and the complex plane. The letters  $E$  and  $F$  are used for complex Banach spaces. The symbols  $E'$  and  $E^*$  denote respectively the algebraic dual and topological dual of  $E$ . We denote by  $U$  a non-empty open subset of  $E$ ; and by  $U_E$  and  $B_E$ , the open and closed unit ball of  $E$ . For a locally convex space  $X$ , we denote by  $X_\beta^*$  and  $X_c^*$ , the topological dual  $X^*$  of  $X$  equipped respectively with the strong topology, *i.e.*, the topology of uniform convergence on all bounded subsets of  $X$ , and the compact open topology.

For each  $m \in \mathbb{N}$ ,  $\mathcal{L}(^m E; F)$  is the Banach space of all continuous  $m$ -linear mappings from  $E$  to  $F$  endowed with its natural sup norm. For  $m=1$ , we write  $\mathcal{L}(E, F)$  for  $\mathcal{L}(^m E; F)$ . A mapping  $P : E \rightarrow F$  is said to be a *continuous  $m$ -homogeneous polynomial* if there exists a continuous  $m$ -linear map  $A \in \mathcal{L}(^m E; F)$  such that

$$P(x) = A(x, \dots, x), \quad x \in E.$$

In this case, we also write  $P = \hat{A}$ . The space of all continuous  $m$ -homogeneous polynomials from  $E$  to  $F$  is denoted by  $\mathcal{P}(^m E; F)$  which is a Banach space endowed with the sup norm. A *continuous polynomial*  $P$  is a mapping from  $E$  into  $F$  which can be represented as a sum  $P = P_0 + P_1 + \cdots + P_k$  with  $P_m \in \mathcal{P}(^m E; F)$  for  $m = 0, 1, \dots, k$ . The vector space of all continuous polynomials from  $E$  into  $F$  is denoted by  $\mathcal{P}(E; F)$ .

A polynomial  $P \in \mathcal{P}(^m E; F)$  is said to be of *finite type* if it is of the form

$$P(x) = \sum_{j=1}^k \phi_j^m(x) y_j, \quad x \in E,$$

where  $\phi_j \in E^*$  and  $y_j \in F$ ,  $1 \leq j \leq k$ . We denote by  $\mathcal{P}_f(^m E; F)$  the space of finite type polynomials from  $E$  into  $F$ . A continuous polynomial  $P$  from  $E$  into  $F$  is said to be of finite type if it has a representation as a sum  $P = P_0 + P_1 + \cdots + P_k$  with  $P_m \in \mathcal{P}_f(^m E; F)$  for  $m = 0, 1, \dots, k$ . The vector space of continuous polynomials of finite type from  $E$  into  $F$  is denoted by  $\mathcal{P}_f(E; F)$ .

A mapping  $f : U \rightarrow F$  is said to be *holomorphic*, if for each  $\xi \in U$ , there exists a ball  $B(\xi, r)$  with center at  $\xi$  and radius  $r > 0$ , contained in  $U$  and a sequence  $\{P_m\}_{m=1}^\infty$  of polynomials with  $P_m \in \mathcal{P}(^m E; F)$ ,  $m \in \mathbb{N}_0$  such that

$$f(x) = \sum_{m=0}^{\infty} P_m(x - \xi), \quad (2.1)$$

where the series converges uniformly for  $x \in B(\xi, r)$ . The series in (2.1) is called the Taylor series of  $f$  at  $\xi$  and in analogy with complex variable case, it is written as

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \hat{d}^m f(\xi)(x - \xi), \quad (2.2)$$

where  $P_m = \frac{1}{m!} \hat{d}^m f(\xi)$ .

The space of all holomorphic mappings from  $U$  to  $F$  is denoted by  $\mathcal{H}(U; F)$ . It is usually endowed with the topology  $\tau_0$  of uniform convergence on compact subsets of  $U$  and  $(\mathcal{H}(U; F), \tau_0)$  is a Fréchet space when  $U$  is an open subset of a finite dimensional Banach space. In case  $U = E$ , the class  $\mathcal{H}(E; F)$  is the space of entire mappings from  $E$  into  $F$ . For  $F = \mathbb{C}$ , we write  $\mathcal{H}(U)$  for  $\mathcal{H}(U; \mathbb{C})$ . We refer to [1], [9], [19] and [22] for notations and various results on infinite dimensional holomorphy.

If  $f \in \mathcal{H}(U; F)$  and  $n \in \mathbb{N}_0$ , we write  $S_n f(x) = \sum_{m=0}^n \frac{1}{m!} \hat{d}^m f(0)(x)$  and  $C_n f(x) = \frac{1}{n+1} \sum_{k=0}^n S_k f(x)$ . It has been shown in [18] that

$$S_n(f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(e^{it}x) D_n(t) dt \quad \text{and} \quad C_n(f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(e^{it}x) K_n(t) dt,$$

where  $D_n(t)$  and  $K_n(t)$  are respectively the Dirichlet and Fejer kernels given as follows:

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^n \cos kt \quad \text{and} \quad K_n(t) = \frac{1}{n+1} \sum_{k=0}^n D_k(t).$$

A subset  $A$  of  $U$  is called  $U$ -bounded if  $A$  is bounded and  $\text{dist}(A, \partial U) > 0$ , where  $\partial U$  denotes the boundary of  $U$ . A mapping  $f$  in  $\mathcal{H}(U; F)$  is of *bounded type* if it maps  $U$ -bounded sets to bounded sets. The space of holomorphic mappings of bounded type is denoted by  $\mathcal{H}_b(U; F)$ . The space  $\mathcal{H}_b(U; F)$  endowed with the topology  $\tau_b$ , the topology of uniform convergence on  $U$ -bounded sets, is a Fréchet space, cf. [1, p. 81]. For  $U = U_E$ , the following result is quoted from [27].

**THEOREM 2.1.** *If  $\{x_n\}$  is a sequence of distinct points in  $U_E$  such that*

$$\lim_{n \rightarrow \infty} \text{dist}(\{x_n\}, \partial U_E) = 0$$

*and  $\{u_n\}$  is a sequence of vectors in  $F$  then there exists  $f \in \mathcal{H}_b(U_E; F)$  such that*

$$f(x_n) = u_n, \quad n = 1, 2, \dots$$

A weight  $w$  on  $U$  is a continuous and strictly positive function satisfying

$$0 < \inf_A w(x) \leq \sup_A w(x) < \infty \quad (2.3)$$

for each  $U$ -bounded set  $A$ . A weight  $w$  defined on an open balanced subset  $U$  of  $E$  is said to be *radial* if  $w(tx) = w(x)$  for all  $x \in U$  and  $t \in \mathbb{C}$ , with  $|t| = 1$ ; and on  $E$  it is said to be *rapidly decreasing* if  $\sup_{x \in E} w(x) \|x\|^m < \infty$  for each  $m \in \mathbb{N}_0$ .

Corresponding to a weight function  $w$ , the weighted space of holomorphic functions is defined as

$$\mathcal{H}_w(U; F) = \left\{ f \in \mathcal{H}(U; F) : \|f\|_w = \sup_{x \in U} w(x) \|f(x)\| < \infty \right\}.$$

The space  $(\mathcal{H}_w(U; F), \|\cdot\|_w)$  is a Banach space and  $B_w$  denotes its closed unit ball. For  $F = \mathbb{C}$ , we write  $\mathcal{H}_w(U) = \mathcal{H}_w(U; \mathbb{C})$ . It can be easily seen that the norm topology  $\tau_{\|\cdot\|_w}$  on  $\mathcal{H}_w(U; F)$  is finer than the topology induced by  $\tau_0$ . In case,  $\mathcal{P}(E) \subset \mathcal{H}_w(U)$ , we have the following result from [12].

**PROPOSITION 2.2.** *The topology  $\tau_{\|\cdot\|_w}$  restricted to  $\mathcal{P}({}^m E)$  coincides with the norm topology.*

Since the closed unit ball  $B_w$  of  $\mathcal{H}_w(U)$  is  $\tau_0$ -compact by the Ascoli's theorem, the predual of  $\mathcal{H}_w(U)$  is given by

$$\mathcal{G}_w(U) = \{\phi \in \mathcal{H}_w(U)'\} : \phi|_{B_w} \text{ is } \tau_0\text{-continuous}\}$$

by the Ng Theorem; cf. [23].

Further, we consider the locally convex topology  $\tau_{bc}$  on  $\mathcal{H}_w(U)$  for which a set  $A \subset \mathcal{H}_w(U)$  is  $\tau_{bc}$  open if and only if  $A \cap B$  is open in  $(B, B|_{\tau_0})$  for each  $\|\cdot\|_w$ -bounded subset  $B$  of  $\mathcal{H}_w(U)$ . Concerning this topology, we have the following result from [25].

**PROPOSITION 2.3.** *Let  $U$  be an open subset of a Banach space  $E$  and  $w$  be a weight on  $U$ . Then*

- (i)  $(\mathcal{H}_w(U), \|\cdot\|_w)$  and  $(\mathcal{H}_w(U), \tau_{bc})$  have the same bounded sets.
- (ii)  $\mathcal{G}_w(U) = (\mathcal{H}_w(U), \tau_{bc})_\beta^*$ .
- (iii)  $(\mathcal{H}_w(U), \tau_{bc}) = \mathcal{G}_w(U)_c^*$ .

An operator  $T$  in  $\mathcal{L}(E; F)$  is said to have a *finite rank* if the range of  $T$  is finite dimensional and, an operator  $T$  in  $\mathcal{L}(E; F)$  is called *compact* if  $T(B_E)$  is a relatively compact subset of  $F$ . We denote by  $\mathcal{F}(E; F)$  and  $\mathcal{K}(E; F)$ , respectively, the space of all finite rank operators and compact operators from  $E$  into  $F$ .

A Banach space  $E$  is said to have the *approximation property* if for every compact set  $K$  of  $E$  and  $\epsilon > 0$ , there exists an operator  $T \in \mathcal{F}(E; E)$  such that

$$\sup_{x \in K} \|T(x) - x\| < \epsilon.$$

The following characterization of the approximation property due to Grothendieck, is given in [16].

THEOREM 2.4. *For a Banach space  $E$ , the following are equivalent:*

- (i)  *$E$  has the approximation property.*
- (ii) *For every Banach space  $F$ ,  $\overline{\mathcal{F}(E; F)}^{\tau_c} = \mathcal{L}(E; F)$ .*
- (iii) *For every Banach space  $F$ ,  $\overline{\mathcal{F}(F; E)}^{\tau_c} = \mathcal{L}(F; E)$ .*
- (iv) *For every Banach space  $F$ ,  $\overline{\mathcal{F}(F; E)}^{\|\cdot\|} = \mathcal{K}(F; E)$ .*

PROPOSITION 2.5. *Let  $E$  be a Banach space. Then  $E^*$  has the approximation property if and only if  $\overline{\mathcal{F}(E; F)}^{\|\cdot\|} = \mathcal{K}(E; F)$ , for every Banach space  $F$ .*

PROPOSITION 2.6. *Let  $E$  be a Banach space with the approximation property. Then each complemented subspace of  $E$  also has the approximation property.*

### 3. LINEARIZATION THEOREM FOR $\mathcal{H}_w(U; F)$ AND ITS APPLICATIONS

In this section, we consider the linearization theorem for  $\mathcal{H}_w(U; F)$  and some of its applications. Let us begin with

THEOREM 3.1. (Linearization Theorem) *For an open subset  $U$  of a Banach space  $E$  and a weight  $w$  on  $U$ , there exists a Banach space  $\mathcal{G}_w(U)$  and a mapping  $\Delta_w \in \mathcal{H}_w(U; \mathcal{G}_w(U))$  with the following property: for each Banach space  $F$  and each mapping  $f \in \mathcal{H}_w(U; F)$ , there is a unique operator  $T_f \in \mathcal{L}(\mathcal{G}_w(U); F)$  such that  $T_f \circ \Delta_w = f$ . The correspondence  $\Psi$  between  $\mathcal{H}_w(U; F)$  and  $\mathcal{L}(\mathcal{G}_w(U); F)$  given by*

$$\Psi(f) = T_f$$

*is an isometric isomorphism. The space  $\mathcal{G}_w(U)$  is uniquely determined up to an isometric isomorphism by these properties.*

*Proof.* Though the proof of this result is similar to the one given in [2], we sketch the same for the sake of completeness.

Let  $B_w$  be the closed unit ball of  $\mathcal{H}_w(U)$ . Then it is  $\tau_0$ -compact by Ascoli's Theorem. Hence by the Ng's Theorem,  $\mathcal{H}_w(U)$  is a dual Banach space, its predual being given by

$$\mathcal{G}_w(U) = \{h \in \mathcal{H}_w(U)' : h|_{B_w} \text{ is } \tau_0\text{-continuous}\}.$$

Further the mapping  $J_U^w : \mathcal{H}_w(U) \rightarrow \mathcal{G}_w(U)^*$ ,  $J_U^w(f) = \hat{f}$  with  $\hat{f}(h) = h(f)$ ,  $f \in \mathcal{H}_w(U)$  and  $h \in \mathcal{G}_w(U)$ , is an isometric isomorphism.

Now define  $\Delta_w : U \rightarrow \mathcal{G}_w(U)$  as  $\Delta_w(x) = \delta_x$ , where  $\delta_x(f) = f(x)$ ,  $f \in \mathcal{H}_w(U)$ .

Since for  $x \in U$  and  $f \in \mathcal{H}_w(U)$ ,  $J_U^w(f) \circ \Delta_w(x) = J_U^w(f)(\delta_x) = f(x)$  and  $J_U^w(\mathcal{H}_w(U)) = \mathcal{G}_w(U)^*$ ,  $\Delta_w$  is weakly holomorphic and hence holomorphic, cf. [1, p.66]. In order to show that  $\Delta_w \in \mathcal{H}_w(U; \mathcal{G}_w(U))$ , fix  $x_0 \in U$ . Then for  $f \in \mathcal{H}_w(U)$ ,  $|\delta_{x_0}(f)| = |f(x_0)| \leq \frac{1}{w(x_0)} \|f\|_w$  implies  $\|\delta_{x_0}\| \leq \frac{1}{w(x_0)}$ . Hence  $\|\Delta_w\|_w = \sup_{x \in U} w(x) \|\delta_x\| \leq 1$ . Consequently,  $\Delta_w \in \mathcal{H}_w(U; \mathcal{G}_w(U))$ .

Corresponding to  $f$  in  $\mathcal{H}_w(U; F)$ , we now define  $T_f$ . For the case  $F = \mathbb{C}$ , define  $T_f = J_U^w(f)$ . Then  $T_f \circ \Delta_w(f) = f$  and  $\|T_f\| = \|f\|_w$ .

In case of an arbitrary Banach space  $F$ , we first define  $T_f : \mathcal{G}_w(U) \rightarrow F^{**}$  as

$$T_f(h)(\phi) = h(\phi \circ f), \quad h \in \mathcal{G}_w(U), \quad \phi \in F^*.$$

Note that  $T_f$  is, indeed,  $F$ -valued; for  $T_f(\delta_x) = f(x) \in F$  and  $\overline{\text{span}}\{\delta_x : x \in U\} = \mathcal{G}_w(U)$ . Further,

$$\|f\|_w = \sup_{x \in U} w(x) \|f(x)\| = \sup_{x \in U} w(x) \|T_f(\delta_x)\| \leq \|T_f\|$$

and

$$\|T_f(h)(\phi)\| \leq \|h\| \|\phi\| \|f\|_w, \quad h \in \mathcal{G}_w(U), \quad \phi \in F^*.$$

Thus  $\|T_f\| = \|f\|_w$  and  $\Psi$  is an isometric isomorphism. ■

*Remark 3.2.* If  $(w\Delta_w)(x) = w(x)\Delta_w(x)$ ,  $x \in U$ , then

$$J_U^w(B_w) = \{(w\Delta_w)(x) : x \in U\}^\circ.$$

Consequently,  $(J_U^w(B_w))^\circ = B_{\mathcal{G}_w(U)} = \bar{\Gamma}\{(w\Delta_w)(x) : x \in U\}$ , where  $\bar{\Gamma}(A)$  denotes the absolutely convex closed hull of  $A$ .

In case the weight  $w$  is given by an entire function  $\gamma$  with positive coefficients, i.e.,  $w(x) = \frac{1}{\gamma(\|x\|)}$ ,  $x \in E$ , we write  $\mathcal{H}_\gamma$  for  $\mathcal{H}_w$ ; and the above linearization theorem takes the following form:

**THEOREM 3.3.** *Let  $\gamma$  be an entire function with positive coefficients. Then for an open subset  $U$  of a Banach space  $E$  and weight  $w$ ,  $w(x) = \frac{1}{\gamma(\|x\|)}$ ,  $x \in U$ , there exists a Banach space  $G_\gamma(U)$  and a mapping  $\Delta_\gamma \in \mathcal{H}_\gamma(U; G_\gamma(U))$ ,  $\|\Delta_\gamma\| = 1$  with the following property: for each Banach space  $F$  and each*

mapping  $f \in \mathcal{H}_\gamma(U; F)$ , there is a unique operator  $T_f \in \mathcal{L}(G_\gamma(U); F)$  such that  $T_f \circ \Delta_\gamma = f$ . The correspondence  $\Psi$  between  $\mathcal{H}_\gamma(U; F)$  and  $\mathcal{L}(G_\gamma(U); F)$  given by

$$\Psi(f) = T_f$$

is an isometric isomorphism. The space  $G_\gamma(U)$  is uniquely determined up to an isometric isomorphism by these properties.

*Proof.* It suffices to prove here that  $\|\Delta_\gamma\| = 1$ . Let  $\gamma(z) = \sum_{n=0}^{\infty} a_n z^n$  with  $a_n > 0$  for each  $n \in \mathbb{N}_0$ . Fix  $x_0 \in E$ . Choose  $\phi \in E^*$  with  $\|\phi\| = 1$  and  $|\phi(x_0)| = \|x_0\|$ . Define  $f : E \rightarrow \mathbb{C}$  as

$$f(x) = \sum_{n=1}^{\infty} a_n \phi^n(x), \quad x \in E.$$

Clearly,  $f \in \mathcal{H}_\gamma(E)$  and  $\|f\|_\gamma \leq 1$ . Since  $|f(x_0)| = \gamma(\|x_0\|)$ , we have

$$\|\delta_{x_0}\| = \sup_{\|h\|_\gamma \leq 1} |h(x_0)| = \gamma(\|x_0\|).$$

Thus  $\|\Delta_\gamma\| = 1$ . ■

Before we consider the applications of the above linearization theorem, let us prove results related to the inclusion of polynomials in the weighted space of holomorphic mappings.

**PROPOSITION 3.4.** *Let  $w$  be a weight defined on an open subset  $U$  of a Banach space  $E$ . Then, for each  $m \in \mathbb{N}$ , the following are equivalent:*

- (a)  $\mathcal{P}^m E; F) \subset \mathcal{H}_w(U; F)$  for each Banach space  $F$ .
- (b)  $\mathcal{P}^m E) \subset \mathcal{H}_w(U)$ .

*Proof.* (a) $\Rightarrow$ (b). Immediate.

(b) $\Rightarrow$ (a). Consider  $Q \in \mathcal{P}^m E; F)$ . For  $x \in U$ , choose  $\phi_x \in F^*$  such that  $\|\phi_x\| = 1$  and  $\phi_x(Q(x)) = \|Q(x)\|$ . Write  $A = \{\phi_x \circ Q : x \in U\}$ . Then  $A$  is a  $\|\cdot\|$ -bounded subset of  $\mathcal{P}^m E$  since  $\|\phi_x \circ Q\| \leq \|Q\|$ . Hence by Proposition 2.2,  $A$  is  $\|\cdot\|_w$ -bounded. Consequently,

$$\|Q\|_w = \sup_{x \in U} w(x) |\phi_x(Q(x))| \leq \sup_{x \in U} \sup_{y \in U} w(y) |\phi_x(Q(y))| < \infty.$$

Thus  $Q \in \mathcal{H}_w(U; F)$  and (a) follows. ■



PROPOSITION 3.5. *Let  $w$  be a weight on an open subset  $U$  of a Banach space  $E$ . Then*

- (a) *If  $U$  is bounded,  $\mathcal{P}(E) \subset \mathcal{H}_w(U)$  if and only if  $w$  is bounded.*
- (b) *For  $U = E$ ,  $\mathcal{P}(E) \subset \mathcal{H}_w(E)$  if and only if  $w$  is rapidly decreasing.*

*Proof.* (a) Since constant functions are in  $\mathcal{P}(E)$ , the proof follows.  
 (b) This is a particular case of a result proved in [12, p.6], by taking the family  $V$  consisting of a single weight. ■

In the remaining part of this section, we consider weights  $w$  defined on an open subset  $U$  of  $E$  so that the space  $\mathcal{P}(E, F)$  is contained in  $\mathcal{H}_w(U, F)$ , for which it suffices to consider the scalar case in view of Proposition 3.4.

PROPOSITION 3.6. *Let  $w$  be a weight defined on an open subset  $U$  of a Banach space  $E$  such that  $\mathcal{P}(E) \subset \mathcal{H}_w(U)$ . Then  $E$  is topologically isomorphic to a complemented subspace of  $\mathcal{G}_w(U)$ .*

*Proof.* Since the inclusion map  $I$  from  $U$  to  $E$  is a member of  $\mathcal{H}_w(U; E)$ , by Theorem 3.1, there exists  $T \in \mathcal{L}(\mathcal{G}_w(U); E)$  and  $\Delta_w \in \mathcal{H}_w(U; \mathcal{G}_w(U))$  such that

$$T \circ \Delta_w(x) = I_w(x) = x, \quad x \in U.$$

Fix  $a \in U$  and write  $S = d^1 \Delta_w(a)$ . Note that  $S \in \mathcal{L}(E; \mathcal{G}_w(U))$ . Further, by Cauchy's integral formula,

$$S(t) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{\Delta_w(a + \zeta t)}{\zeta^2} d\zeta, \quad t \in E,$$

where  $r > 0$  is chosen so that  $\{a + \zeta t : |\zeta| \leq r\} \subset U$ . Now

$$T \circ S(t) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{(a + \zeta t)}{\zeta^2} d\zeta = t, \quad t \in E.$$

This gives  $\|S(t)\| \geq \frac{1}{\|T\|} \|t\|$  and so,  $S$  is injective and  $S^{-1}$  is continuous.

Define  $P = S \circ T$ . Then  $P$  is a projection map from  $\mathcal{G}_w(U)$  into itself. Also  $S(E) = P(\mathcal{G}_w(U))$ . Hence  $S$  is a topological isomorphism between  $E$  and a complemented subspace of  $\mathcal{G}_w(U)$ . ■

For the weight  $w$  as considered in Theorem 3.3, we have

PROPOSITION 3.7. *Let  $\gamma$  be an entire function with positive coefficients and  $t_0$  be a positive real satisfying the equation  $\gamma(t) = t\gamma'(t)$ . Assume that  $U$  is an open subset of a Banach space  $E$  for which  $\{x \in E : \|x\| \leq t_0\} \subset U$ . Then there exists a topological isomorphism  $S$  between  $E$  and a complemented subspace of  $G_\gamma(U)$  with  $\|S\| = \frac{\gamma(t_0)}{t_0}$ .*

*Proof.* Since the weight given by  $\gamma$  is bounded,  $I \in H_\gamma(U; E)$ . By Theorem 3.3, there exists  $T \in \mathcal{L}(G_\gamma(U); E)$  and  $\Delta_\gamma \in \mathcal{H}_\gamma(U; G_\gamma(U))$  such that  $T \circ \Delta_\gamma = I$  and  $\|T\| = \|I\|_\gamma$ . But

$$\|T\| = \|I\|_\gamma = \sup_{x \in U} \frac{\|x\|}{\gamma(\|x\|)} = \frac{t_0}{\gamma(t_0)}. \quad (3.1)$$

Writing  $S$  for  $d^1\Delta_\gamma(0)$ , by Cauchy's inequality, we get

$$\|S\| = \|d^1\Delta_\gamma(0)\| \leq \frac{1}{t_0} \sup_{\|x\|=t_0} \|\Delta_\gamma(x)\| = \frac{1}{t_0} \sup_{\|x\|=t_0} \|\delta_x\| = \frac{\gamma(t_0)}{t_0}. \quad (3.2)$$

Now proceeding as in the proof of Proposition 3.4, we have

$$T \circ S(t) = t, \quad \forall t \in E.$$

Consequently, by (3.1) and (3.2), we get

$$\|t\| = \|T \circ S(t)\| \leq \frac{t_0}{\gamma(t_0)} \|S(t)\| \leq \|t\|, \quad t \in E.$$

Hence,

$$\|S\| = \frac{\gamma(t_0)}{t_0}. \quad \blacksquare$$

Illustrating the above result, we have

EXAMPLE 3.8. Let  $\gamma(z) = e^{\tau z}$ ,  $\tau > 0$ . One can easily find that  $t_0 = \frac{1}{\tau}$ . In this case  $\|I\|_\gamma = \frac{1}{\tau e}$  and  $\|S\| = \tau e$ . If  $\tau = \frac{1}{e}$ ,  $S$  becomes an isometric isomorphism.

For our next result, we make use of the following linearization theorem quoted from [18] and proved by using tensor product techniques for locally convex spaces in [26].

**THEOREM 3.9.** *Let  $E$  be a Banach space and  $m \in \mathbb{N}$ . Then there exists a Banach space  $Q(^mE)$  and a polynomial  $q_m \in \mathcal{P}(^mE; Q(^mE))$  such that for any Banach space  $F$  and each polynomial  $P \in \mathcal{P}(^mE; F)$ , there is a unique operator  $T_P \in \mathcal{L}(Q(^mE); F)$  satisfying  $T_P \circ q_m = P$ . The correspondence  $\Phi : \mathcal{P}(^mE; F) \rightarrow \mathcal{L}(Q(^mE); F)$ ,  $\Phi(P) = T_P$  is an isometric isomorphism and the space  $Q(^mE)$  is uniquely determined up to an isometric isomorphism.*

In the statement of the above result, the space  $Q(^mE)$  is defined as the predual of  $\mathcal{P}(^mE)$ , i.e.,  $\{h \in \mathcal{P}(^mE)' : h|_{B_m} \text{ is } \tau_0\text{-continuous}\}$ , where  $B_m$  is the closed unit ball of  $\mathcal{P}(^mE)$ . The map  $q_m : E \rightarrow Q(^mE)$  is given by  $q_m(x) = \delta_x$ , where  $\delta_x(P) = P(x)$ ,  $P \in \mathcal{P}(^mE)$  or equivalently  $q_m(x) = x \otimes \cdots \otimes x$ , cf. [10, p. 29]. For  $w$  and  $U$  as in Proposition 3.6, we prove

**PROPOSITION 3.10.** *The space  $Q(^mE)$  is topologically isomorphic to a complemented subspace of  $\mathcal{G}_w(U)$ .*

*Proof.* Consider  $q_m \in \mathcal{P}(^mE; Q(^mE))$ . By Theorem 3.1, there exist  $T_m \in \mathcal{L}(\mathcal{G}_w(U); Q(^mE))$  and  $\Delta_w \in \mathcal{H}_w(U; G_w(U))$  such that  $T_m \circ \Delta_w = q_m$ . Let  $S_m$  be the  $m$ -th Taylor series coefficient of  $\Delta_w$  around 'a', i.e.,  $S_m = \frac{1}{m!} \widehat{d}^m \Delta_w(a)$ . As  $S_m \in \mathcal{P}(^mE; \mathcal{G}_w(U))$ , by Theorem 3.9 there exists  $R_m \in \mathcal{L}(Q(^mE); \mathcal{G}_w(U))$  such that  $R_m \circ q_m = S_m$ . Now,

$$T_m \circ R_m \circ q_m = T_m \circ S_m = \frac{1}{m!} \widehat{d}^m (T_m \circ \Delta_w)(a) = \frac{1}{m!} \widehat{d}^m q_m(a).$$

As  $\overline{\text{span}}\{q_m(x) : x \in E\} = Q(^mE)$ , it follows that  $T_m \circ R_m(u) = u$ ,  $u \in Q(^mE)$ . Let  $P_m = R_m \circ T_m$ . Then  $P_m$  is a projection map from  $\mathcal{G}_w(U)$  into itself and  $R_m$  is the topological isomorphism between  $Q(^mE)$  and a complemented subspace of  $\mathcal{G}_w(U)$ . ■

**PROPOSITION 3.11.** *For  $m \in \mathbb{N}$ , there exists a topological isomorphism  $R_m$  between the space  $Q(^mE)$  and a complemented subspace of  $G_\gamma(U)$ , for any open subset  $U$  of  $E$  containing the set  $\{x \in E : \|x\| \leq r_0\}$ ,  $r_0$  being a positive real number satisfying the equation  $r\gamma'(r) - m\gamma(r) = 0$  and  $r_0 > m$ . Further  $\|R_m\| = \frac{\gamma(r_0)}{r_0^m}$ .*

*Proof.* As  $q_m \in H_\gamma(U; Q(^mE))$ , by Theorem 3.3, there exist  $T_m \in \mathcal{L}(G_\gamma(U); Q(^mE))$  and  $\Delta_\gamma \in \mathcal{H}_\gamma(U; G_\gamma(U))$  such that  $T_m \circ \Delta_\gamma = q_m$ . Since  $\sup_{x \in U} \frac{\|x\|^m}{\gamma(\|x\|)} = \frac{r_0^m}{\gamma(r_0)}$ , we have

$$\|q_m\|_\gamma = \|T_m\| = \frac{r_0^m}{\gamma(r_0)}. \quad (3.3)$$

Now by Cauchy's inequality, we get

$$\left\| \frac{1}{m!} \widehat{d}^m \Delta_\gamma(0) \right\| \leq \frac{1}{r_0^m} \sup_{\|x\|=r_0} \|\Delta_\gamma(x)\| = \frac{\gamma(r_0)}{r_0^m}.$$

Continuing as in the proof of the above result, we have

$$T_m \circ R_m(u) = u, \quad u \in Q(^m E). \quad (3.4)$$

By using (3.3) and (3.4), we get

$$\|u\| = \|T_m \circ R_m(u)\| \leq \frac{r_0^m}{\gamma(r_0)} \|R_m(u)\| \leq \|u\|$$

for every  $u \in Q(^m E)$ . Thus  $\|R_m\| = \frac{\gamma(r_0)}{r_0^m}$ . ■

Considering the function given in Example 3.8, we have the following, illustrating the above result

EXAMPLE 3.12. If  $\gamma(z) = e^{\tau z}$ ,  $\tau > 0$ , we find  $r_0 = \frac{m}{\tau}$  and, so  $\|R_m\| = \frac{\tau^m e^m}{m^m}$ .

Also, by using the same argument as in Proposition 3.11, one can easily check

EXAMPLE 3.13. For  $n \in \mathbb{N}$ , define  $w : U_E \rightarrow (0, \infty)$  by  $w(x) = (1 - \|x\|)^n$ ,  $x \in U_E$ . Then

$$\|R_m\| = \left( \frac{n}{m+n} \right)^n$$

for any  $m \in \mathbb{N}$ .

#### 4. THE TOPOLOGY $\tau_{\mathcal{M}}$

In this section we introduce a locally convex topology  $\tau_{\mathcal{M}}$  on  $\mathcal{H}_w(U; F)$  of which the particular cases have been considered in [18] and [25]. For a finite set  $A$  and  $r > 0$ , let us define

$$N(A, r) = \{f \in \mathcal{H}_w(U; F) : \inf_{x \in A} w(x) \sup_{y \in A} \|f(y)\| \leq r\}.$$

Consider the class

$$\mathcal{U} = \left\{ \bigcap_{j=1}^{\infty} N(A_j, r_j) : (A_j) \text{ varies over all sequences of finite subsets of } U \text{ and } (r_j) \text{ varies over all positive sequences diverging to infinity} \right\}$$

It can be easily checked that each member of  $\mathcal{U}$  is balanced, convex and absorbing. Thus it forms a fundamental neighborhood system at 0 for a locally convex topology, which we denote by  $\tau_{\mathcal{M}}$ . Equivalently, this topology is generated by the family

$$\left\{ p_{\bar{\alpha}, \bar{A}} : \bar{\alpha} = (\alpha_j) \in c_0^+, \bar{A} = (A_j), A_j \text{ being finite subset of } U \text{ for each } j \right\}$$

of seminorms given by

$$p_{\bar{\alpha}, \bar{A}}(f) = \sup_{j \in \mathbb{N}} \left( \alpha_j \inf_{x \in A_j} w(x) \sup_{y \in A_j} \|f(y)\| \right).$$

These are the Minkowski functionals of members in  $\mathcal{U}$ . For  $F = \mathbb{C}$ ,  $\tau_{\mathcal{M}} = \tau_{bc}$ , cf. [25, p. 350].

For our results in the sequel, we make use of the following

LEMMA 4.1. *Let  $M$  be a compact subset of  $\mathcal{G}_w(U)$ . Then there exist sequences  $\bar{\alpha} = (\alpha_j) \in c_0^+$  and  $\bar{A} = (A_j)$  of finite subsets of  $U$  such that*

$$M \subset \bar{\Gamma} \left( \bigcup_{j \geq 1} \left\{ \alpha_j \inf_{x \in A_j} w(x) \Delta_w(y) : y \in A_j \right\} \right).$$

*Proof.* Since  $M^\circ$  is a  $\tau_c$ -neighborhood of 0 in  $\mathcal{G}_w(U)^*$ , it is  $\tau_{bc}$ -neighborhood of 0 by Proposition 2.3(iii). Consequently, there exist sequences  $(\alpha_j) \in c_0^+$  and  $\bar{A} = (A_j)$  of finite subsets of  $U$  such that  $\{f \in \mathcal{H}_w(U) : p_{\bar{\alpha}, \bar{A}}(f) \leq 1\} \subset M^\circ$ , where  $M^\circ = \{f \in \mathcal{H}_w(U) : \sup_{u \in M} | \langle f, u \rangle | \leq 1\}$ . Writing  $B = \bigcup_{j \geq 1} \{\alpha_j \inf_{x \in A_j} w(x) \Delta_w(y) : y \in A_j\}$ , we get  $B^\circ \subset M^\circ$ . Therefore, by the bipolar theorem, we have

$$M \subset \bar{\Gamma} \left( \bigcup_{j \geq 1} \left\{ \alpha_j \inf_{x \in A_j} w(x) \Delta_w(y) : y \in A_j \right\} \right). \quad \blacksquare$$

Relating  $\tau_{\mathcal{M}}$  with  $\tau_0$  and  $\tau_{\|\cdot\|_w}$ , and bounded sets with respect to these topologies, we prove

PROPOSITION 4.2. *For a weight  $w$  on an open subset  $U$  of a Banach space  $E$ , the following hold:*

- (i)  $\tau_0 \leq \tau_{\mathcal{M}} \leq \tau_{\|\cdot\|_w}$  on  $\mathcal{H}_w(U; F)$ .
- (ii)  $\tau_{\mathcal{M}}$  and  $\|\cdot\|_w$ -bounded sets are the same.
- (iii)  $\tau_{\mathcal{M}}|_{\mathcal{B}} = \tau_0|_{\mathcal{B}}$  for any  $\|\cdot\|_w$ -bounded set  $\mathcal{B}$ .

*Proof.* (i) Let  $K$  be a compact subset of  $U$ . Then by Lemma 4.1, there exist sequences  $(\alpha_j) \in c_0^+$  and  $\overline{A} = (A_j)$  of finite subsets of  $U$  such that

$$\Delta_w(K) \subset \overline{\Gamma} \left( \bigcup_{j \geq 1} \left\{ \alpha_j \inf_{x \in A_j} w(x) \Delta_w(y) : y \in A_j \right\} \right).$$

Hence, for  $f \in \mathcal{H}_w(U; F)$ , we have

$$\sup_{x \in K} \|f(x)\| = \sup_{x \in K} \|T_f \circ \Delta_w(x)\| \leq p_{\overline{\alpha}, \overline{A}}(f).$$

Thus  $\tau_{\mathcal{M}} \geq \tau_0$  on  $\mathcal{H}_w(U; F)$ . The inequality  $\tau_{\mathcal{M}} \leq \tau_{\|\cdot\|_w}$  clearly holds.

(ii) As every  $\|\cdot\|_w$ -bounded set is  $\tau_{\mathcal{M}}$ -bounded, it suffices to prove the other implication. Assume that there exists a  $\tau_{\mathcal{M}}$ -bounded set  $A$  which is not  $\|\cdot\|_w$ -bounded. Then for each  $k \in \mathbb{N}$ , there exist  $f_k \in A$  such that

$$\|f_k\|_w > k^2.$$

Therefore,  $w(x_k)\|f_k(x_k)\| > k^2$  for some sequence  $\{x_k\} \subset U$ . Consider the  $\tau_{\mathcal{M}}$ -continuous semi-norm  $p$  on  $\mathcal{H}_w(U; F)$  defined by the sequences  $\{\frac{1}{j}\}$  and  $\{x_j\}$  obtained as above, namely

$$p(f) = \sup_{j \in \mathbb{N}} \frac{1}{j} w(x_j) \|f(x_j)\|.$$

Then  $p(\frac{f_k}{k}) > 1$ , for each  $k$ . This contradicts the  $\tau_{\mathcal{M}}$ -boundedness of  $A$  as  $\frac{1}{k} \rightarrow 0$  and  $\{f_k\} \subset A$ , cf. [14, p. 161].

(iii) Let  $\mathcal{B}$  be a bounded set in  $(\mathcal{H}_w(U; F), \|\cdot\|_w)$ . Then there exists a constant  $M > 0$  such that  $\|f\|_w \leq M$ , for every  $f \in \mathcal{B}$ . In order to show that  $\tau_{\mathcal{M}}|\mathcal{B}| \leq \tau_0|\mathcal{B}|$ , consider a  $\tau_{\mathcal{M}}$ -continuous semi-norm  $p$  given by

$$p(f) = \sup_{j \in \mathbb{N}} \left( \alpha_j \inf_{x \in A_j} w(x) \sup_{y \in A_j} \|f(y)\| \right), \quad f \in \mathcal{H}_w(U; F),$$

where  $(\alpha_j) \in c_0^+$  and  $(A_j)$  is a sequence of finite subsets of  $U$ . Fix  $\epsilon > 0$  arbitrarily. Then there exists  $k_0 \in \mathbb{N}$  such that

$$\alpha_j < \frac{\epsilon}{2M}, \quad \forall j > k_0.$$

Write  $K = \bigcup_{j \leq k_0} A_j$ . Then  $K$  is a compact subset of  $U$ . For  $f, g \in \mathcal{B}$ ,

$$p(f - g) < \epsilon \quad \text{whenever} \quad p_K(f - g) < \delta,$$

where

$$\delta = \frac{\epsilon}{\|\bar{\alpha}\|_{\infty} \sup_{1 \leq j \leq k_0} \left( \inf_{x \in A_j} w(x) \right)};$$

indeed

$$\sup_{j \leq k_0} \left( \alpha_j \inf_{x \in A_j} w(x) \sup_{y \in A_j} \|(f - g)(y)\| \right) \leq \|\bar{\alpha}\|_{\infty} \sup_{1 \leq j \leq k_0} \left( \inf_{x \in A_j} w(x) \right) p_K(f - g).$$

This completes the proof as the other implication is obviously true. ■

Proceeding on the lines similar to [25, Remark 3.32], it can be proved that the topology  $\tau_{\mathcal{M}}$  may be strictly finer than  $\tau_0$  on  $\mathcal{H}_w(U; F)$ . However, for the sake of convenience of the reader, we give

EXAMPLE 4.3. Let  $E$  be a Banach space and  $w$  be a bounded weight on  $U_E$ . Assume that  $\tau_{\mathcal{M}} = \tau_0$  on  $\mathcal{H}_w(U_E; F)$ . Choose a sequence  $\{x_n\}$  in  $U_E$  such that  $\|x_n\| \rightarrow 1$  and  $\{u_n\}$  in  $F$  with  $\|u_n\| = n$ ,  $n \in \mathbb{N}$ . Then by Theorem 2.1, there exists a function  $f \in \mathcal{H}_b(U; F)$  such that

$$f(x_n) = \frac{u_n}{w(x_n)}, \quad n \in \mathbb{N}.$$

Since  $\|f\|_w = \sup_{x \in U} w(x) \|f(x)\| > n$  for all  $n \in \mathbb{N}$ ,  $f \notin \mathcal{H}_w(U_E; F)$ . Consequently, the set

$$A = \left\{ \sum_{m=0}^N \frac{1}{m!} \hat{d}^m f(0) : N = 0, 1, 2, \dots \right\}$$

is not  $\|\cdot\|_w$  bounded. But the convergence of the series  $\sum_{m=0}^{\infty} \frac{1}{m!} \hat{d}^m f(0)$  to  $f$  in  $\tau_0$  topology yields that the set  $A$  is  $\tau_0$ -bounded. As  $\tau_{\mathcal{M}}$  and  $\|\cdot\|_w$ -bounded sets are the same by Proposition 4.2(ii), it follows that  $\tau_{\mathcal{M}} \neq \tau_0$ , i.e.,  $\tau_0 < \tau_{\mathcal{M}}$ .

One can easily establish the following observation which we write as

PROPOSITION 4.4. Let  $(A_j)$  be a sequence of finite sets in  $E$  and  $A = \bigcup_{j \in \mathbb{N}} A_j$ . Then  $A$  is bounded if and only if the set  $K = (\bigcup_{j \in \mathbb{N}} \alpha_j A_j) \cup \{0\}$  is compact for each  $\bar{\alpha} = (\alpha_j) \in c_0$ .

*Proof.* Immediate. ■

PROPOSITION 4.5. *Let  $E$  and  $F$  be Banach spaces. For a weight  $w$  on an open subset  $U$  of  $E$  with  $\mathcal{P}(E) \subset \mathcal{H}_w(U)$ ,  $\tau_{\mathcal{M}}$  coincides with  $\tau_0$  on  $\mathcal{P}^m(E; F)$  for each  $m \in \mathbb{N}$ .*

*Proof.* Let  $p$  be a  $\tau_{\mathcal{M}}$ -continuous semi-norm on  $\mathcal{H}_w(U; F)$ . Then there exist sequences  $\bar{\alpha} = (\alpha_j) \in c_0^+$  and  $\bar{A} = (A_j)$  of finite subsets of  $U$  such that

$$p(f) = \sup_{j \in \mathbb{N}} \left( \alpha_j \inf_{x \in A_j} w(x) \sup_{y \in A_j} \|f(y)\| \right), \quad f \in \mathcal{H}_w(U; F).$$

Define  $K = \bigcup_{j \in \mathbb{N}} \{(\alpha_j \inf_{x \in A_j} w(x))^{\frac{1}{m}} y : y \in A_j\} \cup \{0\}$ . For each  $y \in U$ , choose  $\phi_y \in E^*$  with  $\|\phi_y\| = 1$  and  $\phi_y(y) = \|y\|$ . Then the set  $B = \{\phi_y^m : y \in U\}$  is a norm bounded subset of  $\mathcal{P}^m(E)$  and hence  $\|\cdot\|_w$ -bounded by Proposition 2.2. Therefore

$$\sup_{j \in \mathbb{N}} \sup_{y \in A_j} w(y) \|y\|^m \leq \sup_{y \in U} \sup_{x \in U} w(x) \|\phi_y^m(x)\| < \infty.$$

Then by Proposition 4.4,  $K$  is a compact subset of  $E$ . Since

$$p(P) = \sup_{j \in \mathbb{N}} \sup_{y \in A_j} \left\| P \left( (\alpha_j \inf_{x \in A_j} w(x))^{\frac{1}{m}} y \right) \right\| = p_K(P).$$

for any  $P \in \mathcal{P}^m(E; F)$ , the proof follows. ■

Next, we prove

PROPOSITION 4.6. *Let  $E$  and  $F$  be Banach spaces. For a radial weight  $w$  on a balanced open subset  $U$  of  $E$  with  $\mathcal{P}(E) \subset \mathcal{H}_w(U)$ , the space  $\mathcal{P}(E; F)$  is  $\tau_{\mathcal{M}}$ -dense in  $\mathcal{H}_w(U; F)$ .*

*Proof.* Recalling the notations  $S_n(f)$  and  $C_n(f)$ , and their integral representations for  $f \in \mathcal{H}_w(U; F)$  from Section 2, we have

$$\|C_n(f)(x)\| = \left\| \frac{1}{\pi} \int_{-\pi}^{\pi} f(e^{it}x) K_n(t) dt \right\| \leq \sup_{t \in [-\pi, \pi]} \|f(e^{it}x)\|$$

since  $\int_{-\pi}^{\pi} K_n(t) dt = 1$ , cf. [28, p. 45]. Consequently, for each  $n \in \mathbb{N}_0$ ,

$$\|C_n(f)(x)\|_w \leq \sup_{x \in U} w(x) \sup_{|t|=1} \|f(tx)\| = \sup_{x \in U} \sup_{|t|=1} w(tx) \|f(tx)\| \leq \|f\|_w < \infty.$$

Thus, for given  $f \in \mathcal{H}_w(U; F)$ , the set  $\{C_n(f) : n \in \mathbb{N}_0\}$  is  $\|\cdot\|_w$ -bounded in  $\mathcal{H}_w(U; F)$ . As  $C_n f \rightarrow f$  in  $(H(U; F), \tau_0)$ , the result follows by Proposition 4.2(iii). ■



Finally in this section, we consider an analogue of Theorem 3.1 on  $\mathcal{H}_w(U; F)$  when it is equipped with the topology  $\tau_{\mathcal{M}}$ . This result will be useful for our study of approximation properties in the next section. Indeed, we prove

**THEOREM 4.7.** *Let  $E$  and  $F$  be Banach spaces, and  $w$  be a weight on an open subset  $U$  of  $E$ . Then the mapping*

$$\Psi : (\mathcal{H}_w(U; F), \tau_{\mathcal{M}}) \rightarrow (\mathcal{L}(\mathcal{G}_w(U); F), \tau_c)$$

*is a topological isomorphism.*

*Proof.* Let  $M$  be a compact subset of  $\mathcal{G}_w(U)$ . Then by Lemma 4.1, there exist sequences  $(\alpha_j) \in c_0^+$  and  $\bar{A} = (A_j)$  of finite subsets of  $U$  such that

$$M \subset \bar{\Gamma} \left( \bigcup_{j \geq 1} \left\{ \alpha_j \inf_{x \in A_j} w(x) \Delta_w(y) : y \in A_j \right\} \right).$$

Hence for  $f \in \mathcal{H}_w(U; F)$ ,

$$p_M(\Psi(f)) = \sup_{u \in M} \|T_f(u)\| \leq \sup_{j \in \mathbb{N}} \left( \alpha_j \inf_{x \in A_j} w(x) \sup_{y \in A_j} \|f(y)\| \right) = p_{\bar{\alpha}, \bar{A}}(f).$$

Thus  $\Psi$  is  $\tau_{\mathcal{M}} - \tau_c$  continuous.

In order to show the continuity of the inverse map  $\Psi^{-1}$ , let us note that

$$\sup_{j \in \mathbb{N}} \sup_{y \in A_j} \left( \inf_{x \in A_j} w(x) \|\Delta_w(y)\| \right) \leq 1.$$

Hence by Proposition 4.4, the set

$$K = \bar{\Gamma} \left( \bigcup_{j \geq 1} \left\{ \alpha_j \inf_{x \in A_j} w(x) \Delta_w(y) : y \in A_j \right\} \right) \cup \{0\}$$

is a compact subset of  $\mathcal{G}_w(U)$ , which immediately yields the  $\tau_c - \tau_{\mathcal{M}}$  continuity of the inverse mapping  $\Psi^{-1}$ . ■

## 5. THE APPROXIMATION PROPERTIES

This section is devoted to the study of the approximation property for the space  $E$ , the weighted space  $\mathcal{H}_w(U)$  of holomorphic mappings and its predual  $\mathcal{G}_w(U)$ . We write

$$\mathcal{H}_w(U) \otimes F = \{f \in \mathcal{H}_w(U; F) : f \text{ has finite dimensional range}\}$$

and

$$\mathcal{H}_w^c(U; F) = \{f \in \mathcal{H}_w(U; F) : wf \text{ has a relatively compact range}\}.$$

In the next proposition we establish the interplay between the properties of a mapping  $f \in \mathcal{H}_w(U; F)$  and the corresponding operator  $T_f \in \mathcal{L}(\mathcal{G}_w(U); F)$ .

**PROPOSITION 5.1.** *Let  $U$  be an open subset of a Banach space  $E$  and  $w$  be a weight on  $U$ . Then for any Banach space  $F$ ,*

- (a)  $f \in \mathcal{H}_w(U) \otimes F$  if and only if  $T_f \in \mathcal{F}(\mathcal{G}_w(U); F)$ ,
- (b)  $f \in \mathcal{H}_w^c(U; F)$  if and only if  $T_f \in \mathcal{K}(\mathcal{G}_w(U); F)$ .

*Proof.* (a) Note that for  $(g_i)_{i=1}^n \subset \mathcal{H}_w(U)$  and  $(y_i)_{i=1}^n \subset F$ ,

$$f(x) = \sum_{i=1}^n g_i(x)y_i \quad \Leftrightarrow \quad T_f(\delta_x) = \sum_{i=1}^n \langle \delta_x, g_i \rangle y_i$$

for each  $x \in U$ . As  $\mathcal{G}_w(U)^* = \mathcal{H}_w(U)$  and  $\overline{\text{span}}\{\delta_x : x \in U\} = \mathcal{G}_w(U)$ , the result follows.

(b) By Remark 3.2,  $B_{\mathcal{G}_w(U)} = \overline{\Gamma}(w\Delta_w)(U)$ , the result follows from

$$(wf)(U) = T_f((w\Delta_w)(U)) \subset T_f(\overline{\Gamma}(w\Delta_w)(U)) = \overline{\Gamma}((wf)(U)).$$

■

**PROPOSITION 5.2.** *Let  $w$  be a weight on an open subset  $U$  of a Banach space  $E$ . Then  $\overline{\mathcal{F}(\mathcal{G}_w(U); F)}^{\|\cdot\|} = \mathcal{K}(\mathcal{G}_w(U); F)$  if and only if  $\overline{\mathcal{H}_w(U) \otimes F}^{\|\cdot\|_w} = \mathcal{H}_w^c(U; F)$  for each Banach space  $F$ .*

*Proof.* Assume that  $\overline{\mathcal{F}(\mathcal{G}_w(U); F)}^{\|\cdot\|} = \mathcal{K}(\mathcal{G}_w(U); F)$ . Consider  $f \in \mathcal{H}_w^c(U; F)$ . Then  $T_f \in \mathcal{K}(\mathcal{G}_w(U); F)$  by Proposition 5.1(b). Hence there exists a net  $(T_\alpha) \subset \mathcal{F}(\mathcal{G}_w(U); F)$  such that  $T_\alpha \xrightarrow{\|\cdot\|} T_f$ . Now, corresponding to each  $\alpha$ , we have  $f_\alpha \in \mathcal{H}_w(U) \otimes F$  such that  $T_{f_\alpha} = T_\alpha$  by Proposition 5.1(a). Apply Theorem 3.1 to get  $f_\alpha \xrightarrow{\|\cdot\|_w} f$ , thereby proving  $\overline{\mathcal{H}_w(U) \otimes F}^{\|\cdot\|_w} = \mathcal{H}_w^c(U; F)$ . Conversely, for  $T \in \mathcal{K}(\mathcal{G}_w(U); F)$ , there exists  $f \in \mathcal{H}_w^c(U; F)$  such that  $T = T_f$  by Proposition 5.1(b). Then there exists a net  $\{f_\alpha\} \subset \mathcal{H}_w(U) \otimes F$  such that  $f_\alpha \xrightarrow{\|\cdot\|_w} f$ . Thus  $(T_{f_\alpha}) \subset \mathcal{F}(\mathcal{G}_w(U); F)$  by Proposition 5.1(a) and  $T_\alpha \xrightarrow{\|\cdot\|} T_f = T$  by Proposition 3.1. ■

PROPOSITION 5.3. *Let  $w$  be a weight on an open subset  $U$  of a Banach space  $E$ . Then  $\overline{\mathcal{F}(\mathcal{G}_w(U); F)}^{\tau_c} = \mathcal{L}(\mathcal{G}_w(U); F)$  if and only if  $\overline{\mathcal{H}_w(U) \otimes F}^{\tau_M} = \mathcal{H}_w(U; F)$  for each Banach space  $F$ .*

*Proof.* The proof follows analogously by using Theorem 4.7 and Proposition 5.1(b). ■

Characterizing the approximation property for the space  $E$ , we have

THEOREM 5.4. *Let  $E$  be a Banach space. Then for each Banach space  $F$ , the following are equivalent:*

- (i)  $E$  has the approximation property.
- (ii)  $\overline{\mathcal{H}_w(V) \otimes E}^{\tau_M} = \mathcal{H}_w(V; E)$ , for each open subset  $V$  of  $F$  and weight  $w$  on  $V$ .
- (iii)  $\overline{\mathcal{H}_w(V) \otimes E}^{\|\cdot\|_w} = \mathcal{H}_w^c(V; E)$ , for each open subset  $V$  of  $F$  and weight  $w$  on  $V$ .

*Proof.* (i)  $\Rightarrow$  (ii): Assume that  $E$  has the approximation property. Then by Theorem 2.4,  $\overline{\mathcal{F}(\mathcal{G}_w(U); E)}^{\tau_c} = \mathcal{L}(\mathcal{G}_w(U); E)$ . Thus  $\overline{\mathcal{H}_w(V) \otimes E}^{\tau_M} = \mathcal{H}_w(V; E)$  by Proposition 5.3.

(ii)  $\Rightarrow$  (i): We claim that  $\overline{\mathcal{F}(F; E)}^{\tau_c} = \mathcal{L}(F; E)$  for each Banach space  $F$ . Let  $A \in \mathcal{L}(F; E)$ . Applying Proposition 3.4, there exist operators  $S \in \mathcal{L}(F; \mathcal{G}_w(U_F))$  and  $T \in \mathcal{L}(\mathcal{G}_w(U_F); F)$  such that  $T \circ S(y) = y$ ,  $y \in F$ . Since  $\overline{\mathcal{G}_w(U_F)^* \otimes E}^{\tau_M} = \mathcal{H}_w(U_F; E)$  by (ii), in view of Proposition 5.3 there exists a net  $(A_\alpha) \subset \mathcal{F}(\mathcal{G}_w(U_F); E)$  such that  $A_\alpha \xrightarrow{\tau_c} A \circ T$ . Thus  $A_\alpha \circ S \xrightarrow{\tau_c} A \circ T \circ S = A$ . As  $A_\alpha \circ S \subset \mathcal{F}(F; E)$ , our claim holds and (i) follows by Theorem 2.4.

(i)  $\Rightarrow$  (iii): Again using Theorem 2.4,  $\overline{\mathcal{F}(\mathcal{G}_w(U); E)}^{\|\cdot\|} = \mathcal{K}(\mathcal{G}_w(U); E)$  by (i). Therefore  $\overline{\mathcal{H}_w(U) \otimes F}^{\|\cdot\|_w} = \mathcal{H}_w^c(U; F)$  by Proposition 5.2.

(iii)  $\Rightarrow$  (i): Let  $A \in \mathcal{K}(F; E)$  and  $T, S$  be the operators as above. Then  $A \circ T \in \mathcal{K}(\mathcal{G}_w(U_F); E)$ . By hypothesis and Proposition 5.2, there exists a sequence  $(A_n) \subset \mathcal{F}(\mathcal{G}_w(U_F); E)$  such that  $A_n \xrightarrow{\|\cdot\|} A \circ T$ . Thus  $A_n \circ S \xrightarrow{\|\cdot\|} A$  and we have,  $\overline{\mathcal{F}(F; E)}^{\|\cdot\|} = \mathcal{K}(F; E)$ . This proves (i). ■

Next, we characterize the approximation property for the weighted space  $\mathcal{H}_w(U)$ .

**THEOREM 5.5.** *For an open subset  $U$  of a Banach space  $E$ ,  $\mathcal{H}_w(U)$  has the approximation property if and only if  $\mathcal{H}_w(U) \otimes F$  is  $\|\cdot\|_w$ -dense in  $\mathcal{H}_w^c(U; F)$  for each Banach space  $F$ .*

*Proof.* By Proposition 2.5,  $\mathcal{G}_w(U)^*$  has the approximation property if and only if  $\mathcal{F}(\mathcal{G}_w(U); F)$  is  $\|\cdot\|$ -dense in  $\mathcal{K}(\mathcal{G}_w(U); F)$  for each Banach space  $F$ . As  $\mathcal{H}_w(U) = \mathcal{G}_w(U)^*$ , the result follows by Proposition 5.2. ■

We now cite the following known result, cf. [18]; along with the proof for convenience.

**PROPOSITION 5.6.** *If a Banach space  $E$  has the approximation property, then for every Banach space  $F$  and  $m \in \mathbb{N}$ ,  $\mathcal{P}_f(mE; F)^{t_c} = \mathcal{P}(mE; F)$ .*

*Proof.* Let  $P \in \mathcal{P}(mE; F)$ . Then for a compact subset  $K$  of  $E$  and  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $\|P(x) - P(y)\| < \epsilon$  whenever  $x \in K$  and  $y \in E$  with  $\|y - x\| < \delta$ . Since  $E$  has the approximation property, there is a  $T \in \mathcal{F}(E; E)$  such that  $\sup_{x \in K} \|T(x) - x\| < \delta$ . Thus,  $\sup_{x \in K} \|P \circ T(x) - P(x)\| < \epsilon$ . ■

Making use of the above proposition, we finally prove

**THEOREM 5.7.** *Let  $E$  be a Banach space and  $w$  be a radial weight on a balanced open subset  $U$  of  $E$  such that  $\mathcal{H}_w(U)$  contains all the polynomials. Then the following assertions are equivalent:*

- (i)  $E$  has the approximation property.
- (ii)  $\overline{\mathcal{P}_f(E; F)}^{\tau_{\mathcal{M}}} = \mathcal{H}_w(U; F)$  for each Banach space  $F$ .
- (iii)  $\overline{\mathcal{H}_w(U) \otimes F}^{\tau_{\mathcal{M}}} = \mathcal{H}_w(U; F)$  for each Banach space  $F$ .
- (iv)  $\mathcal{G}_w(U)$  has the approximation property.

*Proof.* (i)  $\Rightarrow$  (ii): Let  $p$  be a  $\tau_{\mathcal{M}}$  continuous semi-norm on  $\mathcal{H}_w(U; F)$ . Then for  $f \in \mathcal{H}_w(U; F)$ , there exists  $P \in \mathcal{P}(E; F)$  such that  $p(f - P) < \frac{\epsilon}{2}$  by Proposition 4.6. Let  $P = P_0 + P_1 + \cdots + P_k$ ,  $P_m \in \mathcal{P}(mE; F)$ ,  $0 \leq m \leq k$ . Then by using Proposition 5.6 and Proposition 4.5, there exist  $Q_m$  in  $\mathcal{P}_f(mE; F)$ ,  $0 \leq m \leq k$  such that

$$p(P_m - Q_m) < \frac{\epsilon}{2(k+1)}.$$

Write  $Q = Q_0 + Q_1 + \cdots + Q_k$ . Clearly  $Q \in \mathcal{P}_f(E; F)$  and  $p(f - Q) < \epsilon$ .

(ii)  $\Rightarrow$  (iii): It suffices to prove that  $\mathcal{P}_f(E; F) \subset \mathcal{H}_w(U) \otimes F$ . Consider  $P \in \mathcal{P}_f(E; F)$ . Then there exist  $\phi_j \in E^*$  and  $y_j \in F$ ,  $1 \leq j \leq k$  such that

$$P = \sum_{j=1}^k \phi_j^m \otimes y_j.$$

Now,  $\phi_j^m \in \mathcal{H}_w(U)$  for each  $1 \leq j \leq k$  as  $w$  is bounded. Thus  $P \in \mathcal{H}_w(U) \otimes F$ .

(iii)  $\Rightarrow$  (iv): Note that  $\Delta_w \in \overline{\mathcal{H}_w(U) \otimes \mathcal{G}_w(U)}^{\tau_M}$  by taking  $F = \mathcal{G}_w(U)$  in (iii). Now  $\overline{\mathcal{H}_w(U) \otimes \mathcal{G}_w(U)}^{\tau_M}$  can be identified with  $\overline{\mathcal{F}(\mathcal{G}_w(U); \mathcal{G}_w(U))}^{\tau_c}$  via the map  $\Psi$  by Proposition 5.1(a) and Theorem 4.7. Since  $T_{\Delta_w} \circ \Delta_w = \Delta_w$ , we get  $\Psi(\Delta_w) = I$ , the identity map on  $\mathcal{G}_w(U)$ . Thus  $I \in \overline{\mathcal{F}(\mathcal{G}_w(U); \mathcal{G}_w(U))}^{\tau_c}$ .

(iv)  $\Rightarrow$  (i) follows from Proposition 2.6 and Proposition 3.6. ■

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